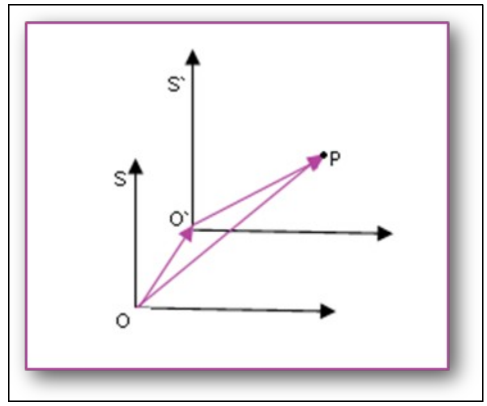
Relative Motion

Frame of Reference

Frame of reference is the coordinate axes with respect to which we determine the position, velocity and acceleration of a particle. These parameters of motion depend upon the frame we have chosen while analyzing the motion.

Translation of Frames

Consider a frame of reference S` which is moving wrt a fixed frame of reference S as shown in the figure:



According to the triangle law of addition of vectors,

$\boldsymbol{OP} = \boldsymbol{OO'} + \boldsymbol{O'P}$

Thus if $\boldsymbol{r}_{PO}$ represents position vector of P wrt O, then

$\boldsymbol{r}_{PO} = \boldsymbol{OP}$ and hence

$\boldsymbol{r}_{PO} = \boldsymbol{r}_{O'O} + \boldsymbol{r}_{PO'}$ ...(i)

Differentiating eqn (i) wrt time,

$\boldsymbol{v}_{PO} = \boldsymbol{v}_{O'O} + \boldsymbol{v}_{PO'}$ ...(ii)

Differentiating eqn (ii) wrt time,

$\boldsymbol{a}_{PO} = \boldsymbol{a}_{O'O} + \boldsymbol{a}_{PO'}$ ...(iii)

The above three equations can be extended for any finite number of frames.

The above equations are valid only if the frames of reference are performing **translatory motion** (i.e. the orientation of the axes does not change wrt space with time).

Also, $\boldsymbol{r}_{PO} = -\boldsymbol{r}_{OP}$

Problem 1: If $\boldsymbol{v}_{AE} = 2i + 3j - 4k$ and $\boldsymbol{v}_{BE} = i + j + k$ and $\boldsymbol{v}_{AC} = -2i + 3j + 4k$ then find out

a) $\boldsymbol{V}_{CE}$

b) $\boldsymbol{V}_{BC}$

c) $\boldsymbol{V}_{BA}$

Some Concepts of Coordinate Geometry

- A line having slope *m* and a y-intercept of *c*, then equation of this line is given by **y = mx + c .**

- A line passing through (x_1,y_1) and (x_2,y_2) has the equation as

$$y - y_1 = [(y_2 - y_1) / (x_2 - x_1)] (x - x_1)$$

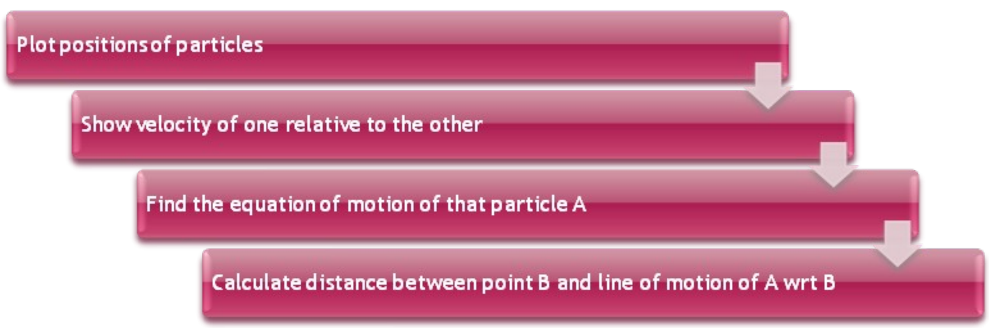
- The perpendicular distance of point $P(x_1,y_1)$ from the line $Ax + By + C = 0$ is given by:

$$L = \left| \frac{Ax1 + By1 + C}{\sqrt{A^2 + B^2}} \right|$$

Concept of Minimum Approach

If initial positions and velocities of the two particles are known, then plot them on a suitable coordinate axes. Show the velocity of any one particle (say, A) wrt the other(B). Next draw the line of motion of A according to angle made by the relative velocity. Find the equation of this line. The perpendicular distance between this line and point B gives the minimum distance between them during the entire motion.

The concepts of coordinate geometry given above must be used here.



Note that if the point B satisfies the equation of motion of A wrt B then the two particles will collide.

Problem 2: Suppose two particles A and B were initially at (1,2) and (9,10) respectively. They were moving with velocities (i+j) and (-3i-4j) respectively. Find out the minimum distance between them during the entire motion.

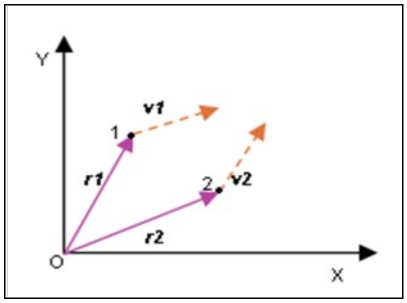
Problem 3: Two particles A and B were initially at (1,2) and (11,12); their initial velocities were (2i+3j) and (-4i-5j) respectively. A had constant acceleration of $3 m/s^2$ towards –ve xaxis while B had an acceleration of $4 m/s^2$ towards y-axis. Find the minimum distance between them during the entire motion. *[Hint: The equations of motion corresponding to constant acceleration can be applied to relative motion provided that the relative acceleration is constant.]*

Problem 4: Two particles A and B start simultaneously from the same point and move in horizontal plane. A has initial velocity *u1* due east and acceleration *a1* due north while B has initial velocity *u2* due North and acceleration *a2* due east. Choose the correct alternative(s):

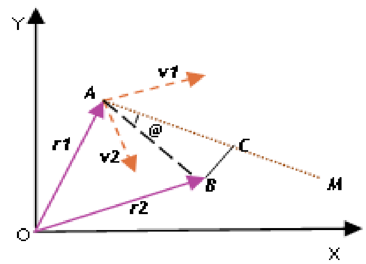
(a) path must intersect at some point.
(b) they must collide at some point.
(c) they will collide only if $a1.u1=a2.u2$.
(d) if $u1>u2$ and $a1<a2$ then the particles will have same speed at some point.

Example 1: Two particles 1 and 2 move with constant velocities ***v1*** and ***v2***. At the initial moment their radius vector are equal to ***r1*** and ***r2***. Find out the minimum distance between 1 and 2 in terms of these two vectors.

Solution 1: Let us plot a hypothetical position as depicted in the question:



Now let us observe the motion of particle 1 wrt 2. In the frame of 2, particle 2 will have zero velocity while particle 1 will have velocity of (***v1*** - ***v2***) wrt 2.



This diagram vividly depicts the motion as seen from the 2-frame. Let dotted line ACM represent the line of motion of A wrt B. This line is in the resultant direction of ***v1*** and –***v2***.

The length of the line BC is the required minimum distance as BC is perpendicular to ACM and BC = AB sin@. From vector theory, we know that |P X Q| = |P||Q| sinθ. Similarly,

$$|\boldsymbol{v}_{AB} \times \boldsymbol{AB}| = |\boldsymbol{v}_{AB}||\boldsymbol{AB}| \sin@$$

Thus, $$BC = |AB| \sin@ = (|\boldsymbol{v}_{AB} \times \boldsymbol{AB}|) / |\boldsymbol{v}_{AB}| \quad \text{...(iv)}$$

Thus minimum distance = |(v1-v2)X(r2-r1)| / |v1-v2| ...(Ans.)

Notes:

- The above equation (iv) can be used as a shortcut formula to find the minimum distance between two particles whose initial positions and velocities are known.
- In the above example, for collision to take place, point B must lie on the line ACM(i.e. the resultant of ***v1*** and –***v2*** must be along vector ***AB***.

$$\frac{v1 - v2}{|v1 - v2|} = \frac{r2 - r1}{|r2 - r1|}$$

Relative Motion in case of two Projectile motions

- The path of a projectile as seen from another projectile is a straight line.
- There is no relative acceleration of one projectile wrt to another projectile.

Thus dealing of two projectile motions is a simple applications of the concepts of relative motions we have learnt so far. The following problem is based on this case of relative motion.

Problem 5: Two particles A and B are projected simultaneously from top of two towers of heights 10m and 15m respectively. The towers are separated by 20m from each other. If A is projected at speed of 20 m/s at an angle of 30° with horizontal, then find out the velocity vector of B wrt earth if they collide just 0.5s after their projection. Also find out the point where they collide.

Non Translatory Change of Frames

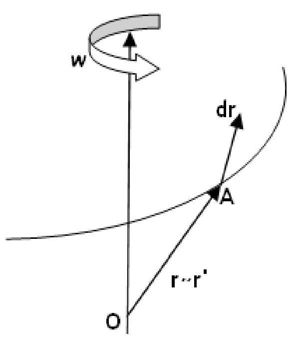
There are two frames of reference S and S'. The velocity ***v*** and the acceleration ***a*** of a point A in S frame is known. We have to find the corresponding values of ***v'*** and ***a'*** wrt S'-frame when S'-frame performs a non translatory motion wrt S-frame.

Here we shall analyze two cases:

1. The S'-frame rotates at the constant angular velocity ***ω*** about an axis which is stationary wrt S-frame.

2. The S'-frame rotates with constant angular velocity ***ω*** about the axis translating with velocity $\boldsymbol{v}_o$ and acceleration $\boldsymbol{a}_o$ relative to S-frame.

CASE-I



Let O be the origin of both S and S' frames. The position vector of point A will then be same in both the frames(i.e. ***r≈r')***.

If the point A is at rest wrt S' frame, then its displacement d***r*** in S frame in time interval dt is only by rotation of ***r*** through angle dΦ. Then the linear displacement of A is associated with angle dΦ by thr relation:

$$|d\boldsymbol{r}| = r \sin\theta \, d\Phi$$

Or, $$d\boldsymbol{r} = d\boldsymbol{\Phi} \times \boldsymbol{r}$$

Only infinitesimal angular displacement can be treated as vectors. The direction of this vector can be judged by Right Hand Thumb Rule. While closing your right hand fist, if the rotation of fingers is same as sense of rotation then the direction of thumb represents the direction of dΦ.

If point A moves at velocity ***v'*** wrt S'-frame, it will cover an additional distance of ***v'dt*** during dt so that

$$d\boldsymbol{r} = \boldsymbol{v}'dt + (d\boldsymbol{\Phi} \times \boldsymbol{r}) \qquad \text{...(v)}$$

Dividing equation (v) by dt, we get

$$\boldsymbol{v} = \boldsymbol{v}' + (\boldsymbol{\omega} \times \boldsymbol{r}) \qquad \text{...(vi)}$$

In equation (vi), ***v*** and ***v'*** are the velocities of point A wrt S and S' frames respectively. According to equation (vi),

$$d\boldsymbol{v} = d\boldsymbol{v}' + (\boldsymbol{\omega} \times d\boldsymbol{r}) \qquad \text{...(vii)}$$

and similar to eqn. (v),

$$\boldsymbol{dv'} = \boldsymbol{a'}dt + (\boldsymbol{d\Phi} \times \boldsymbol{v'}) \qquad \text{...(viii)}$$

Substituting (viii) and (v) into (vii) and dividing by dt,

$$\boldsymbol{a} = \boldsymbol{a'} + 2(\boldsymbol{\omega} \times \boldsymbol{v'}) + (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})) \qquad \text{...(ix)}$$

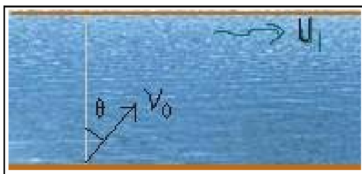
CASE-II

In this case, the S' frame rotates with constant angular velocity ω about the axis translating with the velocity $\boldsymbol{v_0}$ and acceleration $\boldsymbol{a_0}$ wrt the S frame, so just these terms $\boldsymbol{v_0}$ and $\boldsymbol{a_0}$ will be added to equations (vi) and (ix).

$$\boldsymbol{v} = \boldsymbol{v'} + \boldsymbol{v_0} + (\boldsymbol{\omega} \times \boldsymbol{r}) \qquad \text{...(x)}$$

$$\boldsymbol{a} = \boldsymbol{a'} + \boldsymbol{a_0} + 2(\boldsymbol{\omega} \times \boldsymbol{v'}) + (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})) \qquad \text{...(xi)}$$

River-Swimmer System



Upstream and Downstream Swimming

Downstream: The swimmer swims in the direction of river flow.

Upstream : The swimmer swims in direction opposite to the river flow.

$|\boldsymbol{v_{s-r}}| = \boldsymbol{v_0}$ =speed of swimmer wrt river
=speed of swimmer in still water

speed of swimmer wrt river is fixed for a swimmer.

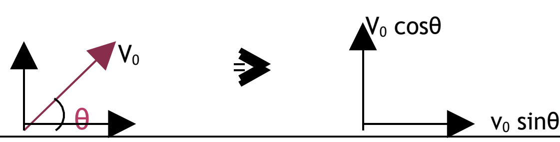
See the figure given alongside, Here v_1 is speed of the river flow and v_0 is the speed of swimmer wrt river.

$\boldsymbol{v_{r-e}} = v_1 \boldsymbol{i}$

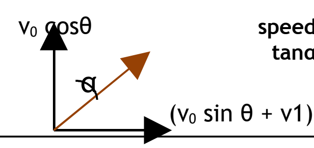
$\boldsymbol{v_{s-r}} = (v_0 \sin\theta)\, \boldsymbol{i} + (v_0 \cos\theta)\, \boldsymbol{j}$

$\boldsymbol{v_{s-e}} = (v_0 \sin\theta + v_1)\, \boldsymbol{i} + (v_0 \cos\theta)\, \boldsymbol{j}$

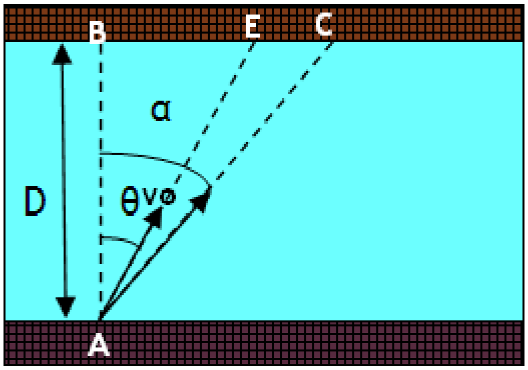
Vector Diagram wrt the River



Vector Diagram wrt the earth



speed = $\sqrt{((v_1+v_0\cos\theta)^2+(v_0\cos\theta)^2)}$
tanα =(v1+v0 sinθ)/(v0cos θ)



Refer to the figure given alongside,

The swimmer starts from point A aiming to reach E but due to the flow of river, he will directly reach at C.The width of the river is D.The velocity component of swimmer along the width of the river is $v_0 \cos\theta$.Thus the time(t) taken to cross the river will be:

$$t = \frac{D}{v_0 \cos\theta}$$

for minimum time to be taken, $\cos\theta$ must be 1 which implies that θ must be 0˚.*Thus, if a swimmer wants to cross the river by taking minimum time then he must start swimming in direction of **AB** and the minimum time will be (= D/ v_0).*

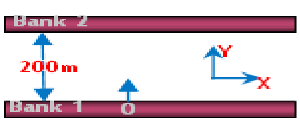
Drift: Drift is the displacement of the swimmer along river flow wrt earth. In the above figure, BC is the drift.

The velocity component along river flow wrt earth is ($v_0 \sin\theta + v_1$) and the time taken is $t=D/ v_0 \cos\theta$.
The drift, $BC=(v_0 \sin\theta + v_1)(D/ v_0 \cos\theta)$

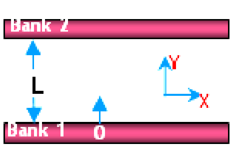
No Drift!
Suppose we want to find which direction must swimmer aim such that there is no drift at all. This will happen when velocity component along river flow wrt earth is zero. i.e. $v_0 \sin\theta + v_1 = 0$ or $\sin\theta = (-v_1/v_0)$
Thus for no drift, swimmer must take angle (θ) opposite to sense we have taken in the figure and this new θ must be equal to $\sin^{-1}(v_1/v_0)$.

Problem 6: Consider a river having width 200m. A person starts from one bank and crosses the river in minimum time of 10s. In this process, drift made by him is 100m. Find out the time taken by the swimmer to cross the same river with no drift.

Problem 7: The flow speed of water in a river varies with the distance from its bank 1 as v = (2y) m/s,the river is flowing along +ve X-direction. A swimmer whose swimming speed is 5 m/s(wrt water) enters the river at point O and swims along +ve Y axis. The drift of the swimmer when he reaches bank 2 is _________m .



Problem 8: The velocity of flow of stream between two parallel banks, bank 1 and bank 2 is varying uniformly from 0 to v over the width of the river. A boat starts rowing with constant speed u (wrt stream) from bank 1 to reach bank 2. The direction of flow of river is along +X-axis.



a) If the boat is moving in such a way that wrt observer on shore it is always moving along a perpendicular line to banks, then what is the angle made by boat's bow with perpendicular line as a function of y(Take u>v) ?

b) For the situation mentioned above, what will be the time taken by boat to cross the river?

c) If u=(3/4)v then up to what value of y the observer will see the boat is moving along a line perpendicular to banks?

Rain-Man System, Aeroplane-Wind System, Rain-Wind System, etc. can be analysed and dealt in somewhat similar manner as we have done with the River-Swimmer system.

Notes...